

Characterization of Completely k -Magic Regular Graphs

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Abstract

Let $k \in \mathbb{N}$ and $c \in \mathbb{Z}_k$, where $\mathbb{Z}_1 = \mathbb{Z}$. A graph $G = (V(G), E(G))$ is said to be c -sum k -magic if there is a labeling $\ell : E(G) \rightarrow \mathbb{Z}_k \setminus \{0\}$ such that $\sum_{u \in N(v)} \ell(uv) \equiv c \pmod{k}$ for every vertex v of G , where $N(v)$ is the neighborhood of v in G . We say that G is completely k -magic whenever it is c -sum k -magic for every $c \in \mathbb{Z}_k$. In this paper, we characterize all completely k -magic regular graphs.

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1 Introduction

Let $G = (V(G), E(G))$ be a finite, simple (unless otherwise stated) graph with vertex set $V(G)$ and edge set $E(G)$. A *factor* of G is a subgraph H with $V(H) = V(G)$. In particular, if a factor H of G is h -regular, then we

say that H is an h -factor of G . An h -factorization of G is a partition of $E(G)$ into disjoint h -factors. If such factorization of G exists, then we say that G is h -factorable.

The following theorem is attributed to Petersen [8], which we state using the versions of Akiyama and Kano [2] and Wang and Hu [12].

Theorem 1.1 ([2, Theorem 3.1], [8], [12, Theorem 10]). *Let G be a $2r$ -regular connected general graph (not necessarily simple), where $r \geq 1$. Then G is 2-factorable, and it has a $2k$ -factor for every k , $1 \leq k \leq r$. Moreover, if G is of even order, then it is r -factorable.*

A graph G is λ -edge connected if it remains connected whenever fewer than λ edges are removed.

Theorem 1.2. [7] *Let r and k be integers such that $1 \leq k < r$, and G be a λ -edge connected r -regular general graph, where $\lambda \geq 1$. If one of the following conditions holds:*

- (1) *r is even, k is odd, $|G|$ is even, and $\frac{r}{\lambda} \leq k \leq r(1 - \frac{1}{\lambda})$,*
- (2) *r is odd, k is even, and $2 \leq k \leq r(1 - \frac{1}{\lambda})$, or*
- (3) *r and k are both odd and $\frac{r}{\lambda} \leq k$,*

then G has a k -regular factor.

Let A be a non-trivial Abelian group written additively. A finite simple graph $G = (V(G), E(G))$ is said to be A -magic if there exists an edge labeling $\ell : E(G) \rightarrow A \setminus \{0\}$ such that the induced vertex labeling $\ell^+ : V(G) \rightarrow A$, defined by $\ell^+(v) = \sum_{uv \in E(G)} \ell(uv)$, is a constant map. If $c \in A$ and $\ell^+(v) = c$ for all $v \in V(G)$, then we call c is a *magic sum* of G .

Let k be a positive integer. If G is \mathbb{Z}_k -magic graph, then we say that G is k -magic. Here, $\mathbb{Z}_1 = \mathbb{Z}$ the group of integers, and $\mathbb{Z}_k = \{0, 1, 2, \dots, k-1\}$ the group of integers modulo $k \geq 2$. In particular, if G is k -magic with magic sum c , then we say that G is c -sum k -magic. If G is c -sum k -magic for all $c \in \mathbb{Z}_k$, then it is said to be *completely k -magic*. The set of all magic sums $c \in \mathbb{Z}_k$ of G is the *sum spectrum of G with respect to k* and is denoted by $\Sigma_k(G)$. If $c = 0$, then we say that G is *zero-sum k -magic*. The *null set* of G , denoted by $N(G)$, is the set of all positive integers k such that G is a zero-sum k -magic graph.

Remark 1.3. *If $c \in \mathbb{Z}_k$ and ℓ is a c -sum k -magic labeling of G , then the labeling ℓ' , defined by $\ell'(e) = k - \ell(e)$, is a $(k - c)$ -sum k -magic labeling of G .*

Remark 1.4. *Any 2-magic graph is not completely 2-magic.*

The concept of A -magic graphs is due to Sedlacek [10]. Over the years, many papers have been published in connection with magic graphs. To name a few, Akbari, Rahmati, and Zare [1] and Choi, Georges, and Mauro [4] investigated the zero-sum k -magic labelings and null sets of regular graphs. Dong and Wang [5] solved affirmatively a conjecture posed in [1] on the existence of a zero-sum 3-magic labeling of 5-regular graphs. Salehi [9] determined the integer-magic spectra of certain classes of cycle-related graphs. Shiu and Low [11] analyzed the group-magic property for complete n -partite graphs and composition graphs with deleted edges.

Using the term “index set,” Wang and Hu [12] initially studied the concept of completely k -magic graphs. They gave a partial list of completely 1-magic regular graphs. Eniego and Garces [6] completely added the remaining cases in this list. They also presented the sum spectra of some regular graphs that are not completely k -magic.

Theorem 1.5 ([1, Theorem 13]). *Let G be an r -regular graph, where $r \geq 3$ and $r \neq 5$. If r is even, then $N(G) = \mathbb{N}$; otherwise, $\mathbb{N} \setminus \{2, 4\} \subseteq N(G)$.*

Theorem 1.6 ([5, Theorem 2.1]). *Every 5-regular graph admits a zero-sum 3-magic labeling.*

Theorem 1.7 ([6, Theorem 3.3]). *Let $n \geq 3$ and $k \geq 3$ be integers, and C_n the cycle with n vertices.*

- (1) *If n is even, then C_n is completely k -magic for all k .*
- (2) *If n is odd, then C_n is not completely k -magic for any k . Moreover, we have*

$$\Sigma_k(C_n) = \begin{cases} \mathbb{Z}_k \setminus \{0\} & \text{if } k \text{ is odd,} \\ \{0, 2, \dots, k-2\} & \text{if } k \text{ is even.} \end{cases}$$

Theorem 1.8 ([6, Lemma 3.4]). *Let $k \geq 4$ be an even integer. Then there exists no k -magic graph of odd order that is completely k -magic. In particular, if c is a magic sum of a k -magic graph of odd order, then c must be even.*

Theorem 1.9 ([6, Theorem 3.6]). *Let $k, r \geq 3$ be integers, and G an r -regular graph. If $\gcd(r, k) = 1$, then $\{1, 2, \dots, k-1\} \subseteq \Sigma_k(G)$.*

Theorem 1.10 ([6, Theorem 3.7]). *Let G be a zero-sum k -magic r -regular graph, where $k \geq 3$ and $r \geq 3$. If G has a 1-factor, then G is completely k -magic.*

Theorem 1.11 ([12, Theorem 13], [6, Theorem 2.1]). *Let G be an r -regular graph of order n . Then*

$$\Sigma_1(G) = \begin{cases} \mathbb{Z} \setminus \{0\} & \text{if } r = 1, \\ \mathbb{Z} & \text{if } r = 2 \text{ and } G \text{ contains even cycles only,} \\ 2\mathbb{Z} \setminus \{0\} & \text{if } r = 2 \text{ and } G \text{ contains an odd cycle,} \\ 2\mathbb{Z} & \text{if } r \geq 3, r \text{ even, and } n \text{ odd,} \\ \mathbb{Z} & \text{if } r \geq 3 \text{ and } n \text{ even,} \end{cases}$$

where $2\mathbb{Z}$ is the set of all even integers.

With Remark 1.4 and Theorem 1.11, it remains to characterize all completely k -magic regular graphs for $k \geq 3$. This characterization is the main theorem of this paper, which we state as follows.

Theorem 1.12 (Main Theorem). *Let $r \geq 2$ and $k \geq 3$ be integers, and G an r -regular graph of order $n \geq 3$. Then G is completely k -magic if and only if one of the following properties holds:*

- (1) $k \geq 3$, $r = 2$, and G contains even cycles only,
- (2) $k \geq 5$ and $r \geq 3$ odd,
- (3) $k \geq 5$, $r \geq 4$ even, and n even,
- (4) $k \geq 5$ odd, $r \geq 4$ even, and n odd,
- (5) $k = 4$, $r \geq 3$, n even, and G zero-sum 4-magic, or
- (6) $k = 3$ and any one of the following conditions holds:
 - (i) $r \not\equiv 0 \pmod{3}$,
 - (ii) $r \equiv 0 \pmod{6}$, or
 - (iii) $r \equiv 0 \pmod{3}$, r odd, and G has a factor H such that $d_H(v) \equiv 1 \pmod{3}$ for all $v \in V(H)$.

For convenience, we assume that all graphs to be considered are finite and simple (unless otherwise stated). We also write \mathbb{Z}_k^* to mean $\mathbb{Z}_k \setminus \{0\}$. For terms that are not defined in this paper, see [3].

2 Proof of the Main Theorem

We divide the proof into several results.

It is not difficult to see that if G is 1-regular, then $\Sigma_k(G) = \mathbb{Z}_k^*$. For 2-regular graphs, the following remark is a consequence of Theorem 1.7.

Remark 2.1. *Let $k \geq 3$ and G a 2-regular graph. If G has an odd cycle, then*

$$\Sigma_k(G) = \begin{cases} \mathbb{Z}_k^* & \text{if } k \text{ is odd} \\ \{0, 2, \dots, k-2\} & \text{if } k \text{ is even.} \end{cases}$$

Otherwise, we have $\Sigma_k(G) = \mathbb{Z}_k$.

Clearly, if G is 1-factorable, then G is completely k -magic. The following theorem considers regular graphs that has a factor that is completely k -magic.

Theorem 2.2. *Let $r \geq 2$, $2 \leq h \leq r$, $k \neq 2$, and G an r -regular graph. If G has an h -factor that is completely k -magic, then G is completely k -magic.*

Proof. The case when $h = r$ is trivial, so we assume $h < r$. Let H be an h -factor of G that is completely k -magic. Let $\alpha = c - (r - h) \pmod{k}$ and f_α be an α -sum k -magic labeling of H for each $c \in \mathbb{Z}_k$.

Define $\ell_c : E(G) \rightarrow \mathbb{Z}_k^*$ by

$$\ell_c(e) = \begin{cases} f_\alpha(e) & \text{if } e \in E(H) \\ 1 & \text{if } e \in E(G \setminus H). \end{cases}$$

The sum of the labels of the edges incident to each vertex of G is $c - (r - h) + (r - h) \equiv c \pmod{k}$. Thus, ℓ_c is a c -sum k -magic labeling of G for each $c \in \mathbb{Z}_k$. Hence, G is completely k -magic. \square

The following construction will be useful in the proofs of our succeeding results.

Remark 2.3. *Let G be an r -regular graph with $E(G) = \{e_1, e_2, e_3, \dots, e_m\}$, where $r \geq 1$. Then we can construct a graph G' (with parallel edges) such that $V(G') = V(G)$ and $E(G') = E(G) \cup \{e'_1, e'_2, e'_3, \dots, e'_m\}$, where e'_i is a duplicate edge of e_i in G for each i (that is, edges e_i and e'_i have the same end vertices). By Theorem 1.1, G' has a $2h$ -factor H' for each h , $1 \leq h \leq r$. Also, $G' \setminus H'$ is a $(2r - 2h)$ -factor of G' obtained by removing the edges of H' from G' .*

Theorem 2.4. *Let G be a 5-regular graph. Then $\mathbb{N} \setminus \{2, 4\} \subseteq N(G)$.*

Proof. We know from Theorem 1.11 and Theorem 1.6 that $1, 3 \in N(G)$. For $k \geq 5$, we consider two cases.

CASE 1. Suppose $k \geq 5$ and $k \neq 8$. Using the construction and notation described in Remark 2.3, let H' be a 2-factor of G' . Then $G' \setminus H'$ is an 8-factor of G' .

Define a zero-sum k -magic labeling ℓ' on G' by

$$\ell'(e) = \begin{cases} k-4 & \text{if } e \in E(H') \\ 1 & \text{if } e \in E(G' \setminus H'). \end{cases}$$

Note that the labeling ℓ on G defined by $\ell(e_i) = \ell'(e_i) + \ell'(e'_i)$ for $e_i \in E(G)$ is a zero-sum k -magic labeling on G .

CASE 2. Suppose $k = 8$. Using again the construction in Remark 2.3, let H' be a 4-factor of G' . Then $G' \setminus H'$ is a 6-factor of G' .

Define a zero-sum labeling ℓ' on G' by

$$\ell'(e) = \begin{cases} 2 & \text{if } e \in E(H') \\ 4 & \text{if } e \in E(G' \setminus H'). \end{cases}$$

Observe that the labeling ℓ on G defined by $\ell(e_i) = \frac{1}{2}[\ell'(e_i) + \ell'(e'_i)]$ for $e_i \in E(G)$ is a zero-sum 8-magic labeling on G .

Therefore, $\mathbb{N} \setminus \{2, 4\} \subseteq N(G)$. \square

Note that an odd-regular graph may not be zero-sum 4-magic. It was remarked in [1, Remark 10] that an odd-regular graph G is not zero-sum 4-magic if G has a vertex such that every edge incident to it is a cut-edge.

Theorem 2.5. *Let G be an r -regular graph, where $r \geq 3$ is odd and $k \geq 5$. Then G is completely k -magic.*

Proof. We know from Theorems 1.5 and 2.4 that $0 \in \Sigma_k(G)$. Let $E(G) = \{e_1, e_2, e_3, \dots, e_m\}$. As constructed in Remark 2.3, let H' and $G' \setminus H'$ be a 2-factor and $(2r-2)$ -factor of G' , respectively. We consider two cases.

CASE 1. Suppose $r \equiv 1 \pmod{k}$. Then $\gcd(r, k) = 1$. By Theorem 1.9, G is completely k -magic.

CASE 2. Suppose $r \not\equiv 1 \pmod{k}$. Assume $\gcd(r, k) = d$ so that $r = ad$ and $k = bd$ for some positive integers a and b . Note that, since r is odd, d is also odd. We consider two sub-cases.

SUB-CASE 2.1. Suppose $k \geq 5$ is odd. Then b is odd.

For each $c \in \mathbb{Z}_k^* \setminus \{k-b, k-2b\}$, define $\ell'_c : E(G') \rightarrow \mathbb{Z}_k^*$ by

$$\ell'_c(e) = \begin{cases} x & \text{if } e \in E(H') \\ \frac{1}{2}(k+b) & \text{if } e \in E(G' \setminus H'), \end{cases}$$

where $x = \frac{1}{2}(b+c)$ if c is odd, and $x = \frac{1}{2}(b+c+k)$ if c is even. Observe that ℓ'_c is a c -sum k -magic labeling of G' for each $c \neq 0$.

For each $c \notin \{0, k-b, k-2b\}$, define $\ell_c : E(G) \rightarrow \mathbb{Z}_k^*$ by $\ell_c(e_i) = \ell'_c(e_i) + \ell'_c(e'_i)$ for $1 \leq i \leq m$. Since ℓ'_c is a c -sum k -magic labeling of G' , ℓ_c is a c -sum k -magic labeling of G for each $c \in \mathbb{Z}_k^* \setminus \{k-b, k-2b\}$.

If $k \neq 3b$, then, by Remark 1.3, $k-b, k-2b \in \Sigma_k(G)$. If $k = 3b$, it is enough to show that $k-2b \in \Sigma_k(G)$. To do that, we provide a different labeling using a different set of factors of G' . Let J' and $G' \setminus J'$ be a 4-factor and $(2r-4)$ -factor of G' respectively. In addition, we let $J' = J'_1 \cup J'_2$, where J'_1 and J'_2 are 2-factors of J' .

Define $\ell' : E(G') \rightarrow \mathbb{Z}_k^*$ by

$$\ell'(e) = \begin{cases} \frac{b+1}{2} & \text{if } e \in E(J'_1) \\ \frac{b-1}{2} & \text{if } e \in E(J'_2) \\ b & \text{if } e \in E(G' \setminus J'). \end{cases}$$

Since $k = 3b$, $d = 3$ and $r = 3a$. Thus, the magic sum in G' is given by $2(\frac{b+1}{2}) + 2(\frac{b-1}{2}) + b(2r-4) \equiv -2b \pmod{k}$. Define $\ell : E(G) \rightarrow \mathbb{Z}_k^*$ by $\ell(e_i) = \ell'(e_i) + \ell'(e'_i)$ for $1 \leq i \leq m$. Note that ℓ is also a $(k-2b)$ -sum k -magic labeling of G .

SUB-CASE 2.2. Suppose $k \geq 6$ is even. Then b is even.

By labeling all the edges of G with $\frac{1}{2}k$, we see that $\frac{1}{2}k \in \Sigma_k(G)$.

Suppose $r-1 \equiv \frac{1}{2}k \pmod{k}$. For each $c \in \mathbb{Z}_k^* \setminus \{k-1, \frac{1}{2}k\}$, define $\ell'_c : E(G') \rightarrow \mathbb{Z}_k^*$ by

$$\ell'_c(e) = \begin{cases} c & \text{if } e \in E(H') \\ 1 & \text{if } e \in E(G' \setminus H'). \end{cases}$$

Observe that the sum of the labels of the edges incident to each vertex in G' is $2(r-1) + 2c \equiv 2c \pmod{k}$. Using a similar argument as in Sub-Case 2.1, it can be shown that G is also e -sum k -magic for all even $e \neq 0$. Thus, we are left to show that G is c -sum k -magic as well for all odd c .

For each odd $c \neq k-1$, define $\ell_c : E(G) \rightarrow \mathbb{Z}_k^*$ by $\ell_c(e_i) = \frac{1}{2}[\ell'_c(e_i) + \ell'_c(e'_i)]$ for each i , $1 \leq i \leq m$. Note that, since ℓ'_c is a $2c$ -sum k -magic labeling of G' , ℓ_c is a c -sum k -magic labeling of G for each odd $c \neq k-1$. Again, by Remark 1.3, we see that $k-1 \in \Sigma_k(G)$.

Suppose $r-1 \equiv r_0 \pmod{k}$, where $r_0 \neq \frac{1}{2}k$. For each $c \in \mathbb{Z}_k^* \setminus \{r_0, r_0 + \frac{1}{2}k, r_0 - 1\}$, define $\ell'_c : E(G') \rightarrow \mathbb{Z}_k^*$ by

$$\ell'_c(e) = \begin{cases} c - r_0 & \text{if } e \in E(H') \\ 1 & \text{if } e \in E(G' \setminus H'). \end{cases}$$

Observe that the sum of the labels of the edges incident to each vertex in G' is $2r_0 + 2c - 2r_0 \equiv 2c \pmod{k}$. As in Sub-Case 2.1, it can be shown that G is also even-sum k -magic. So again, we are left to show that G is odd-sum k -magic.

As what we did earlier, for each odd $c \neq r_0 - 1$ (and, possibly, $r_0 + \frac{1}{2}k$), define $\ell_c : E(G) \rightarrow \mathbb{Z}_k^*$ by $\ell_c(e_i) = \frac{1}{2}[\ell'_c(e_i) + \ell'_c(e'_i)]$ for all i , $1 \leq i \leq m$. Since ℓ'_c is a $2c$ -sum k -magic labeling of G' , ℓ_c is a c -sum k -magic labeling of G for each odd $c \neq r_0 - 1$ (and, possibly, $r_0 + \frac{1}{2}k$). If $r_0 - 1$ and $r_0 + \frac{1}{2}k$ are not inverses, then, by Remark 1.3, $\mathbb{Z}_k^* \subset \Sigma_k(G)$.

If $r_0 - 1$ and $r_0 + \frac{1}{2}k$ are inverses, then it is enough to show that $r_0 - 1 \in \Sigma_k(G)$. Define ℓ' on G' by

$$\ell'(e) = \begin{cases} k-1 & \text{if } e \in E(H') \\ 1 & \text{if } e \in E(G' \setminus H'). \end{cases}$$

Note that the magic sum using ℓ' is $2r_0 - 2$. Define ℓ on G by $\ell(e_i) = \frac{1}{2}[\ell'(e_i) + \ell'(e'_i)]$ for $e_i \in E(G)$. Clearly, ℓ is an $(r_0 - 1)$ -sum k -magic labeling on G . Thus, by Remark 1.3, $r_0 + \frac{1}{2}k \in \Sigma_k(G)$, and so $\mathbb{Z}_k^* \subset \Sigma_k(G)$.

In any case, G is completely k -magic. \square

Theorem 2.6. *Let $k \geq 5$ and G a $2r$ -regular graph of order $n \geq 3$, where $r \geq 2$.*

- (1) *If n is even, then G is completely k -magic.*
- (2) *If n is odd, then*
 - (i) *G is completely k -magic if k is odd, and*
 - (ii) *$\Sigma_k(G) = \{0, 2, 4, \dots, k-2\}$ if k is even.*

Proof. Let $E(G) = \{e_1, e_2, e_3, \dots, e_m\}$. By Theorem 1.5, G is zero-sum k -magic.

(1) Suppose $r = 2$. To prove the theorem, we only show that $\mathbb{Z}_k^* \subset \Sigma_k(G)$. We consider two cases.

CASE 1. Suppose k is odd. Then $\gcd(4, k) = 1$. By Theorem 1.9, $\mathbb{Z}_k^* \subseteq \Sigma_k(G)$.

CASE 2. Suppose k is even. It is not difficult to see that, being 4-regular, G is 2-edge connected. By Remark 2.3, we can construct G' so that G' is a 4-edge-connected 8-regular graph. By Theorem 1.2, G' has a 3-factor, say H' . Let $G' \setminus H'$ be the 5-factor of G' obtained by removing the edges of H' from G' .

SUB-CASE 2.1. Let $k = 2d$, d even. For each $c \in \mathbb{Z}_k^* \setminus \{\frac{1}{2}k, \frac{1}{4}k\}$, define $f_c : E(G') \rightarrow \mathbb{Z}_k^*$ by

$$f_c(e) = \begin{cases} 2c & \text{if } e \in E(H') \\ k - c & \text{if } e \in E(G' \setminus H'). \end{cases}$$

Observe that the sum of the labels of the edges incident to each of the vertices in G' is equal to $5(k - c) + 3(2c) \equiv c \pmod{k}$. This shows that f_c is a c -sum k -magic labeling of G' for all $c \neq 0, \frac{1}{2}k, \frac{1}{4}k$. By Remark 1.3, $\frac{1}{4}k \in \Sigma_k(G')$.

For each $c \in \mathbb{Z}_k^* \setminus \{\frac{1}{2}k, \frac{1}{4}k\}$, define $\ell_c : E(G) \rightarrow \mathbb{Z}_k^*$ by $\ell_c(e_i) = f_c(e_i) + f_c(e'_i)$ for all i , $1 \leq i \leq m$. Clearly, ℓ_c is a c -sum k -magic labeling of G for each $c \in \mathbb{Z}_k^* \setminus \{\frac{1}{2}k, \frac{1}{4}k\}$. By Remark 1.3, we see that $\mathbb{Z}_k^* \setminus \{\frac{1}{2}k\} \subset \Sigma_k(G)$.

By Theorem 1.1, G is 2-factorable. Let G_1 and G_2 be the two 2-factors of G . Label the edges in G_1 with d and the edges in G_2 with $\frac{1}{2}(k - d)$. This shows that $d = \frac{1}{2}k \in \Sigma_k(G)$.

SUB-CASE 2.2. Let $k = 2d$, $d \geq 3$ odd. Observe that, for $c \neq 0, \frac{1}{2}k$, the labeling ℓ_c in Sub-Case 2.1 is a c -sum k -magic labeling of G . To complete the proof, we only need to show that $\frac{1}{2}k \in \Sigma_k(G)$.

Let $d \neq 3$ and 9 . We give a labeling for the factors of G' defined above (namely, H' and $G' \setminus H'$) and the 2-factors of G (namely, G_1 and G_2) to show that G is d -sum k -magic.

Let $f : E(G) \rightarrow \mathbb{Z}_k^*$ be defined by

$$f(e) = \begin{cases} d + 1 & \text{if } e \in E(G_1) \\ \frac{1}{2}(k - d - 1) & \text{if } e \in E(G_2). \end{cases}$$

Clearly, f is $(d + 1)$ -sum k -magic labeling of G .

Let $g' : E(G') \rightarrow \mathbb{Z}_k^*$ be defined by

$$g'(e) = \begin{cases} k - 2 & \text{if } e \in E(H') \\ 1 & \text{if } e \in E(G' \setminus H'). \end{cases}$$

Define also $g : E(G) \rightarrow \mathbb{Z}_k^*$ by $g(e_i) = g'(e_i) + g'(e'_i)$ for all i , $1 \leq i \leq m$. Note that g' is a $(k - 1)$ -sum k -magic labeling of G' , so g is a $(k - 1)$ -sum k -magic labeling of G .

Finally, define $\ell : E(G) \rightarrow \mathbb{Z}_k^*$ by $\ell(e) = f(e) + g(e)$ for all $e \in E(G)$. Since f and g are $(d + 1)$ -sum and $(k - 1)$ -sum k -magic labeling of G respectively, ℓ is a d -sum k -magic labeling of G .

Suppose $d = 3$ or 9 . Define $g' : E(G') \rightarrow \mathbb{Z}_k^*$ be defined by

$$g'(e) = \begin{cases} 2x & \text{if } e \in E(H') \\ 1 & \text{if } e \in E(G' \setminus H'), \end{cases}$$

where $x = 1$ if $d = 3$, and $x = 3$ if $d = 9$. Note that g' is a 5-sum k -magic labeling of G' . Define a labeling g on G by $g(e_i) = g'(e_i) + g'(e'_i) + 1$ for all i , $1 \leq i \leq m$. Note that g is a d -sum k -magic labeling on G . Thus, $d = \frac{1}{2}k \in \Sigma_k(G)$, and so G is completely k -magic.

Suppose $r \geq 3$ is odd. By Theorem 1.1, G is r -factorable. By Theorem 2.5, the r -factors of G are completely k -magic for all $k \geq 5$. Thus, by Theorem 2.2, G is also completely k -magic.

If $r \geq 4$ is even, then, by Theorem 1.1, G has a 6-factor, say H . Using the case for r is odd, H is completely k -magic. Thus, by Theorem 2.2, G is also completely k -magic.

(2(i)) By Theorem 1.1, G is 2-factorable. Let G_1, G_2, \dots, G_r be the 2-factors of G . If k is odd, then, by Remark 2.1, $\mathbb{Z}_k^* \subseteq \sum_k(G_i)$ for all i , $1 \leq i \leq r$. For each i and $c \in \mathbb{Z}_k^*$, let ℓ_c^i be a c -sum k -magic labeling of G_i . We consider two cases.

CASE 1. Suppose $r \equiv 1 \pmod{k}$. For each $c \in \mathbb{Z}_k^*$, define $\ell_c : E(G) \rightarrow \mathbb{Z}_k^*$ by

$$\ell_c(e) = \begin{cases} \ell_c^1(e) & \text{if } e \in E(G_1) \\ \ell_1^i(e) & \text{if } e \in E(G_i) \text{ for some } i = 2, 3, \dots, r. \end{cases}$$

Note that ℓ_c is a c -sum k -magic labeling of G for all $c \neq 0$.

CASE 2. Suppose $r \not\equiv 1 \pmod{k}$. For each $c \in \mathbb{Z}_k^* \setminus \{r-1 \pmod{k}\}$, define $l_c : E(G) \rightarrow \mathbb{Z}_k^*$ by

$$l_c(e) = \begin{cases} l_{c-x}^1(e) & \text{if } e \in E(G_1) \\ l_1^i(e) & \text{if } e \in E(G_i) \text{ for some } i = 2, 3, \dots, r, \end{cases}$$

where $x \equiv r-1 \pmod{k}$. The sum of the labels of the edges incident to each vertex is $c \pmod{k}$. Thus, G is c -sum k -magic for each $c \neq x$. By Remark 1.3, G is x -sum k -magic since G is $(k-x)$ -sum k -magic. In this case, G is completely k -magic.

(2(ii)) This follows from Remark 2.1, Lemma 1.8, and Theorem 2.2. \square

Theorem 2.7. *Let $r \geq 3$, and G a zero-sum 4-magic r -regular graph. Then*

- (1) *If the order of G is even, then G is completely 4-magic.*
- (2) *If the order of G is odd, then $\Sigma_4(G) = \{0, 2\}$.*

Proof. (1) Suppose the order of G is even. We consider two cases.

CASE 1. Suppose r is odd. Clearly, $\gcd(r, 4) = 1$, and so, by Theorem 1.9, $\mathbb{Z}_4^* \subset \Sigma_4(G)$.

CASE 2. Suppose $r = 2x$ for some $x \geq 2$. We consider two sub-cases.

SUB-CASE 2.1. Suppose x is odd. By Theorem 1.1, G is x -factorable. Let G_1 and G_2 be the two edge-disjoint x -factors of G . From Case 1, \mathbb{Z}_4^* is a subset of both $\Sigma_4(G_1)$ and $\Sigma_4(G_2)$. Thus, we have $\mathbb{Z}_4^* \subset \Sigma_4(G)$.

SUB-CASE 2.2. Suppose x is even. If $x = 2$, then, as observed previously, G is 2-edge connected. By Remark 2.3, we can construct G' so that G' is a 4-edge-connected 8-regular graph. By Theorem 1.2, G' has a 5-factor, say H' . Let $G' \setminus H'$ be the 3-factor of G' obtained by removing the edges of H' from G' .

Define $f : E(G') \rightarrow \mathbb{Z}_k^*$ by

$$f(e) = \begin{cases} 1 & \text{if } e \in E(H') \\ 3 & \text{if } e \in E(G' \setminus H'). \end{cases}$$

Observe that the sum of the labels of the edges incident to each of the vertices in G' is $5(1) + 3(3) \equiv 2 \pmod{4}$. Define $\ell : E(G) \rightarrow \mathbb{Z}_k^*$ by $\ell(e_i) = \frac{1}{2}[f(e_i) + f(e'_i)]$ for all i , $1 \leq i \leq m$. Clearly, ℓ is a 1-sum k -magic labeling of G . By Remark 1.3, we see that G is 3-sum 4-magic as well.

To show that G is 2-sum 4-magic, we consider a different labeling for G . By Theorem 1.1, G is 2-factorable. Let G_1 and G_2 be the two 2-factors of G . Label the edges in G_1 with 2 and the edges in G_2 with 1. This shows that G is 2-sum 4-magic.

Suppose $x \geq 4$. By Theorem 1.1, G has a $2y$ -factor for each $1 \leq y \leq x$. In particular, G has a 6-factor, say H . Let $G \setminus H$ be the $(2x - 6)$ -factor of G obtained by removing the edges of H from G . By Sub-Case 2.1, \mathbb{Z}_4^* is a subset of both $\Sigma_4(H)$ and $\Sigma_4(G \setminus H)$. Again, it is not difficult to see that $\mathbb{Z}_4^* \subset \Sigma_4(G)$.

(2) Suppose the order of G is odd. In this case, we only consider $2r$ -regular graphs, $r \geq 2$. By Lemma 1.8, G is not c -sum 4-magic for both $c = 1$ and $c = 3$. To show that G is 2-sum 4-magic, observe that, by Theorem 1.1, G is 2-factorable. Let G_1, G_2, \dots, G_r be the r edge-disjoint 2-factors of G . Label the edges in G_1 with 1, and label the edges in G_i with 2 for all $i \neq 1$. This labeling shows that G is 2-sum 4-magic. By assumption, $0 \in \Sigma_4(G)$. Thus, $\Sigma_4(G) = \{0, 2\}$. \square

The last theorem to complete the proof of the Main Theorem characterizes all completely 3-magic r -regular graphs, where $r \geq 3$.

Theorem 2.8. *Let G be an r -regular graph, where $r \geq 3$.*

- (1) *If $r \not\equiv 0 \pmod{3}$ or $r \equiv 0 \pmod{6}$, then G is completely 3-magic.*
- (2) *If $r \equiv 0 \pmod{3}$ and r odd, then G is completely 3-magic if and only if G has a factor H such that $d_H(v) \equiv 1 \pmod{3}$ for all $v \in V(H)$.*

Proof. (1) Suppose $r \equiv 1 \pmod{3}$. By Theorem 1.5, G is zero-sum 3-magic. By labeling the edges of G with 1, the sum of the labels of the edges incident to each vertex is $r \equiv 1 \pmod{3}$, and G is 1-sum 3-magic. By Remark 1.3, G is also 2-sum 3-magic.

Suppose $r \equiv 2 \pmod{3}$. By Theorems 1.5 and 1.6, G is zero-sum 3-magic. By using Remark 1.3 again and by labeling the edges of G with 2, it follows that G is 1-sum and 2-sum 3-magic.

Let $r = 2(3y)$, $y \geq 1$. By Theorem 1.1, G has a $2z$ -factor for each $1 \leq z \leq 3y$. Let H be a 4-factor of G and $G \setminus H$ be the $(6y - 4)$ -factor of G . As considered above, both H and $G \setminus H$ are completely 3-magic. Thus, by Theorem 2.2, G is completely 3-magic.

(2) Suppose G has a factor H such that $d_H(v) \equiv 1 \pmod{3}$ for all $v \in V(H)$. Denote by $G \setminus H$ the factor of G obtained by removing the edges in G that is an edge in H . Since G is $3x$ -regular, $x \geq 1$, we have $d_{G \setminus H}(v) \equiv 2 \pmod{3}$ for all $v \in V(G \setminus H)$. Label each edge in H with 2 and each edge in $G \setminus H$ with 1. Note that the sum of the labels of the edges incident to each vertex in G is $2(1) + 1(2) \equiv 1 \pmod{3}$. This shows that G is 1-sum 3-magic.

Conversely, suppose G is completely 3-magic. It follows that G is 1-sum 3-magic. Since G is $3x$ -regular (where $x \geq 1$), for any 1-sum 3-magic labeling of G and for each vertex $v \in V(G)$, v must be incident to p edges (where $p \equiv 1 \pmod{3}$) with label 2 and q edges (where $q \equiv 2 \pmod{3}$) with label 1. Let H' be a subgraph of G such that an edge $e \in E(H')$ if and only if the label of e is 2. Clearly, H' is a factor of G and that $d_{H'}(v) \equiv 1 \pmod{3}$ for all $v \in V(H')$. \square

Corollary 2.9. *Let G be an r -regular graph, where $r \geq 3$. Then G is completely 3-magic if and only if one of the following conditions holds:*

- (1) $r \not\equiv 0 \pmod{3}$,
- (2) $r \equiv 0 \pmod{6}$, or
- (3) $r \equiv 0 \pmod{3}$, r odd, and G has a factor H such that $d_H(v) \equiv 1 \pmod{3}$ for all $v \in V(H)$.

References

- [1] S. Akbari, F. Rahmati, and S. Zare, Zero-sum magic labelings and null sets of regular graphs, *The Electronic Journal of Combinatorics* **21**(2) (2014), #P2.17.

- [2] J. Akiyama and M. Kano, *Factors and Factorizations of Graphs*, Springer-Verlag, 2011.
- [3] J.A. Bondy and U.S.R. Murty, *Graph Theory*, Springer, 2008.
- [4] J.-O. Choi, J.P. Georges, and D. Mauro, On zero-sum \mathbb{Z}_k -magic labelings of 3-regular graphs, *Graphs and Combinatorics* **29** (2013), 387-398.
- [5] G. Dong and N. Wang, A conjecture on zero-sum 3-magic labeling of 5-regular graphs, arXiv:1406.6870v1, 2014.
- [6] A.A. Eniego and I.J.L. Garces, On completely k -magic regular graphs, *Applied Mathematical Sciences (Ruse)* **103** (2015), 5139-5148.
- [7] T. Gallai, On factorisation of graphs, *Acta Mathematica Academiae Scientiarum Hungarica* **1**(1) (1950), 133-153.
- [8] J. Petersen, Die Theorie der regulären graphs, *Acta Mathematica* (15) (1891), 193-220.
- [9] E. Salehi, Integer-magic spectra of cycle-related graphs, *Iranian Journal of Mathematical Sciences and Informatics* **2** (2006), 53-63.
- [10] J. Sedlacek, On magic graphs, *Mathematica Slovaca* **26** (1976), 329-335.
- [11] W.C. Shiu and R.M. Low, Group-magic labelings of graphs with deleted edges, *Australasian Journal of Combinatorics* **57** (2013), 3-19.
- [12] T.-M. Wang and S.-W. Hu, Constant sum flows in regular graphs, In *Frontiers in Algorithmics and Algorithmic Aspects in Information and Management*, M. Attalah, X.-Y. Li, and B. Zhu (Editors), Springer Berlin Heidelberg (2011), 168-175.